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1991 J. Phys. A: Math. Gen. 24 5587

(http://iopscience.iop.org/0305-4470/24/23/023)

J. Phys. A: Math. Gen. 24 (1991) 5587-5603. Printed in the UK

Corner transfer matrix eigenstates for the six-vertex model

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Received 11 April 1991, in final form 24 July 1991

Abstract. Eigenstates of the corner transfer matrix (CTM) for the six-vertex model are constructed, and their relation to the Bethe ansatz eigenstates of the XXZ Hamiltonian is discussed. In the ferromagnetic regime ($\Delta > 1$), eigenstates with any finite number of overturned arrows are constructed. They are related to the Bethe ansatz states by Fourier transformation over spectral (rapidity) variables. This exhibits the role of the CTM as a lattice boost operator. For the antiferromagnetic regime, the relation between CTM eigenstates and those of the XXZ Hamiltonian is complicated by the filling of the sea and associated Bethe ansatz integral equations. For this case we construct CTM eigenstates by first showing that the Hamiltonian ground state is also an eigenstate of the CTM. Other CTM eigenstates are constructed from excited eigenstates of the Hamiltonian by Fourier transforming over the rapidity of each dressed excitation. We discuss the relationship between CTM eigenstates and a Heisenberg algebra of bosonic oscillators.

1. Introduction

The corner transfer matrix (CTM) method, developed some time ago in a series of papers by Baxter [1, 2], has proven to be a powerful technique for calculating certain physical quantities in exactly solvable lattice statistical mechanics models. The results of such calculations are typically exact expressions for order parameters (e.g. the spontaneous magnetization of the Ising or eight-vertex model [2], or the local height probabilities for the ABF restricted SOS models [3]). The characteristic feature of these results is that they are given in terms of infinite product expressions with elegant automorphic properties [4]. The form of these expressions follows from the extremely simple eigenvalue structure of the CTM. The spectrum of CTM eigenvalues for the eight-vertex, hard hexagon and RSOS models was first derived by Baxter using an indirect argument which avoided an explicit construction of CTM eigenstates. For the calculation of order parameters (one-point functions) this is sufficient, since the transformation which diagonalizes the CTM commutes with the spin at the origin. Choosing this spin to be the one whose expectation value is computed, the one-point functions may be obtained from the CTM eigenvalue spectrum alone without knowing the structure of the eigenstates. The extreme elegance of these calculations for order parameters raises the hope that the CTM approach may also provide new insight into the generally unsolved problem of computing correlation functions for exactly solvable models. If one hopes to apply the CTM formalism to the calculation of higher n-point correlation functions, information about the CTM eigenstates as well as its eigenvalues is needed.

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0305-4470/91/235587+17$03.50

Another reason for investigating the structure of CTM eigenstates is associated with the apparently fundamental role of infinite dimensional Virasoro [5] and Kac–Moody [6] algebras in the expressions obtained for order parameters. It was observed that the expressions obtained by Andrews, Baxter and Forrester for the local height probabilities of the RSOS models could be written as simple combinations of Virasoro and principally specialized Kac–Moody characters [4]. This provided strong evidence at the character level that the formulae for local height probabilities are connected with a GKO [12] coset construction of irreducible Virasoro modules. This is particularly remarkable, since this structure is exhibited by the exact results for non-critical models, which are not conformally symmetric. These observations appear to promise new insight into the structure of non-critical integrable systems based on the geometrical and analytic structures associated with these infinite dimensional Lie algebras. Further progress along these lines would seem to require an understanding of the structure of the CTM and its eigenstates. The role of the CTM in a lattice Virasoro algebra has been discussed previously [7]. This algebra generates diffeomorphisms of the spectral parameter (lattice rapidity) space, and the CTM itself provides the one parameter subgroup corresponding to uniform rotations of the spectral parameter (boosts) [8]. Thus the eigenstates of the CTM are expected to fall into irreducible Verma modules which carry representations of the lattice Virasoro algebra. To construct the CTM eigenstates we will diagonalize the Virasoro generator L_0, which is related to the CTM by

$$A(\alpha)=\exp(-\alpha L_0) \tag{1.1}$$

where $A(\alpha)$ is the CTM, and α is the spectral parameter. For the six-vertex model, L_0 is given explicitly by the first moment of the XXZ spin chain density [9]. This is the same spin chain density whose zeroth moment H_{XXZ} (i.e. the XXZ Heisenberg Hamiltonian) appears in the expansion of the row-to-row transfer matrix. (Similar statements hold for the XYZ density and the eight-vertex model.) The boost property of L_0 can be expressed at the operator level in terms of the commutator between L_0 and the monodromy matrix $\mathscr{T}(\alpha)$ [8] whose components are used to construct the row-to-row transfer matrix and its Bethe eigenstates in the algebraic Bethe ansatz (quantum inverse) method. In this form it is seen to follow directly from the Yang–Baxter equations (see section 3).

In this paper, we describe an explicit construction of CTM eigenstates for the case of the six-vertex model. This problem has recently been investigated by Davies [10], who has constructed the CTM eigenstates for the case of $N=1$ and 2 overturned arrows. The one- and two-body states we construct in section 2 are equivalent to those obtained by Davies. However, our approach exposes a close connection with the structure of the Bethe ansatz which is used to construct eigenstates of the row-to-row transfer matrix (RTM) and XXZ spin chain Hamiltonian. It is this connection which allows us to construct eigenstates with any number of overturned arrows. (The connection between the CTM and Hamiltonian, or RTM, eigenstates was first pointed out in [7].) We consider first the ferromagnetic regime of the XXZ chain, $\Delta>1$, for which few-body excitations are constructed by a Bethe ansatz for the overturned spins on the ferromagnetic (all spins up) ground state. For this case, the CTM acting on the Bethe eigenstates induces a uniform shift of the rapidities labelling each overturned spin. The eigenstates of the CTM are thus obtained by Fourier transforming over each of the rapidity variables in the state. In the limit $\Delta\to\infty$ (zero temperature limit of the six-vertex model), the rapidity becomes equal to the momentum, and the integer or half-integer eigenvalues of the boost generator L_0 are associated with the site number of the overturned arrow.

In this case, a mode with eigenvalue $l+\frac{1}{2}$ corresponds to an overturned spin at site l. For $1<\Delta<\infty$, the eigenvalues are still half-integers, but now the eigenvalue $l+\frac{1}{2}>0$ corresponds to a down-arrow wavefunction which is peaked near site l and falls off exponentially for large site number j. The wavefunction also vanishes identically for site numbers $j<0$. The exponential fall-off of the wavefunction for large j is determined by λ where $\Delta=\cosh\lambda$.

In the antiferromagnetic regime $\Delta<-1$, the Hamiltonian eigenfunctions for the ground state and low-lying excited states are much more complicated, describing particle excitations in a filled Dirac sea. In the infinite volume limit, the sea is described by a density function in rapidity space which, for a given excitation, is determined by solving a Bethe ansatz integral equation. For this case we find that the action of the CTM on the Hamiltonian ground state no longer induces a uniform rapidity shift of the bare modes. Rather it induces a non-uniform shift which leaves the vacuum density unchanged. Thus the Hamiltonian ground state is invariant under CTM boosts, and it is therefore also an eigenstate of L_0 with zero eigenvalue. Although it is difficult to demonstrate this result directly from the Bethe ansatz expressions, we have found an alternative derivation which is valid to all orders of perturbation theory around the antiferromagnetic, anisotropic limit $\Delta\to-\infty$. The proof is presumably reliable throughout the antiferromagnetic regime $\Delta<-1$, but cannot be used in its present form to discuss the disordered regime $|\Delta|<1$.

Having established the fact that the Hamiltonian ground state is also an eigenstate of L_0, we then consider the low-lying excited states of H_{XXZ}. Here the physical 'particles' consist of bare excitations (n-strings and holes) along with comoving fluctuations in the vacuum density (whose form is given by another Bethe ansatz integral equation). The action of L_0 is again that of a physical boost operator, shifting the rapidity of both the bare excitation and its comoving density fluctuation.

2. Few-body eigenstates

Both the Hamiltonian and the operator L_0 (1.1) of the XXZ chain commute with the z-component of the total spin, that is the number of overturned arrows. Hence the construction of eigenstates can be achieved considering each sector separately. In the sectors of one and two particles this construction has been done by straightforward solution of the Schrödinger equation: for the Hamiltonian one obtains plane waves for $n=1$ and superpositions of these with appropriate phase factors reflecting the scattering phaseshift for $n=2$. For the operator L_0 the corresponding eigenstates can be constructed following the same lines [10]: the amplitudes of the $n=1$ states are given in terms of hyperbolic polynomials leading to localized states in the ordered regime, for the $n=2$ states the amplitudes can be written in terms of the $n=1$ results. Unfortunately, however, this approach is not easily generalized to $n>2$ and thus makes the study of the particularly interesting *antiferromagnetic* regime (where n is of the order of the number of lattice sites) impossible.

For the Hamiltonian the solution of the general n case is provided within the framework of the algebraic Bethe ansatz and the quantum inverse scattering method (QISM): here we construct commuting generators that allow us to build any state starting from the ferromagnetic vacuum.

In this section we will reconsider the construction of eigenstates of L_0 with one and two overturned arrows. Rather than solving the corresponding eigenvalue problem directly, however, we make use of an intimate relation between the eigenstates of L_0 and those of the XXZ Hamiltonian. Together with the results of the QISM this relation allows us to extend this method to sectors with arbitrary n as will be discussed in the following sections.

We start with a short review of the quantum inverse formalism for the six-vertex model. The elementary vertex is defined in terms of the R-matrix:

$$R_j(\alpha)=\begin{pmatrix}\sinh(\lambda/2-\mathrm{i}\alpha S_j^z) & \sinh\lambda S_j^- \\ \sinh\lambda S_j^+ & \sinh(\lambda/2+\mathrm{i}\alpha S_j^z)\end{pmatrix}. \tag{2.1}$$

Here the elements of R_j are labelled by horizontal arrows in an auxiliary space while the operators S_j^β are spin-operators acting on the vertical arrows at site j. With R_j the monodromy matrix is given as

$$\mathcal{T}(\alpha)=\prod_j R_j(\alpha)=\begin{pmatrix}A(\alpha) & B(\alpha)\\ C(\alpha) & D(\alpha)\end{pmatrix}. \tag{2.2}$$

Note that each of the elements of $\mathcal{T}(\alpha)$ is an operator acting on a chain of spins with $S=\frac{1}{2}$. The diagonal elements form the generating functional for integrals of motion for the spin-chain: the row-to-row transfer matrix is $t(\alpha)=\mathrm{trace}\,\mathcal{T}(\alpha)$ satisfying $[t(\alpha), t(\beta)]=0$. The off-diagonal elements generate the spectrum of $t(\alpha)$ (see below).

It is easily seen that the R-matrix reduces to a permutation operator at the point $\alpha=\mathrm{i}\lambda$. Consequently, the monodromy matrix (2.2) reduces to a simple shift operator and one can define a spin-chain Hamiltonian on a lattice with periodic boundary conditions (i.e. $\boldsymbol{S}_{j+N}=\boldsymbol{S}_j$) by

$$H_{XXZ}=-\mathrm{i}\frac{\partial}{\partial\alpha}\ln(t(\alpha)/\sinh^N[\tfrac{1}{2}(\lambda-\mathrm{i}\alpha)])_{\alpha=\mathrm{i}\lambda}=\sum_j H(j,j+1) \tag{2.3}$$

with

$$\begin{aligned}H(j,j+1)&=\frac{1}{\sinh\lambda}\left(\tfrac{1}{2}(S_j^+S_{j+1}^-+S_j^-S_{j+1}^+)-\Delta(S_j^zS_{j+1}^z-\tfrac{1}{4})\right)\\ \Delta&=\cosh\lambda.\end{aligned} \tag{2.4}$$

Clearly, the state with all spins up $|0\rangle=|\uparrow\uparrow\uparrow\ldots\rangle$ is an eigenstate of the transfer matrix with eigenvalue

$$t_0(\alpha)=(\sinh^N[\tfrac{1}{2}(\lambda-\mathrm{i}\alpha)]+\sinh^N[\tfrac{1}{2}(\lambda+\mathrm{i}\alpha)]). \tag{2.5}$$

Starting from this state more eigenstates can be constructed using the properties of the monodromy matrix. One finds that (note that $[B(\alpha), B(\beta)]=0$)

$$|\alpha_1\ldots\alpha_M\rangle=B(\alpha_1)\ldots B(\alpha_M)|0\rangle \tag{2.6}$$

is an eigenstate of $t(\alpha)$ provided that the α_j satisfy the Bethe ansatz equations:

$$\left[\frac{\sinh\frac{1}{2}(\lambda+\mathrm{i}\alpha_j)}{\sinh\frac{1}{2}(\lambda-\mathrm{i}\alpha_j)}\right]^N=\prod_{k\neq j}\frac{\sinh(\frac{1}{2}\mathrm{i}(\alpha_k-\alpha_j)-\lambda)}{\sinh(\frac{1}{2}\mathrm{i}(\alpha_k-\alpha_j)+\lambda)}. \tag{2.7}$$

Each of these states is an eigenstate of the Hamiltonian (2.3) with eigenvalue

$$E=\sum_{j=1}^M\frac{\cosh\lambda-\cos p_j}{\sinh\lambda} \tag{2.8}$$

where the p_j are the momenta of the elementary excitations (magnons), related to the rapidities α_j by

$$e^{ip} = \frac{wz-1}{w-z} \qquad w = e^{\lambda} \qquad z = e^{-i\alpha}. \tag{2.9}$$

The eigenstate corresponding to a single magnon with momentum $p(z)$ on the lattice with periodic boundary conditions is (up to an overall phase absorbed into the definition of the origin of the lattice)

$$|z\rangle = B(z)|0\rangle = \left(\frac{z}{w}\right)^{1/2} \frac{w^2-1}{w-z} \sum_j \left(\frac{wz-1}{w-z}\right)^{j-1} S_j^-|0\rangle. \tag{2.10}$$

Note that as long as we consider an infinite chain with only a finite number of spins overturned, the Bethe ansatz equations do not impose any restrictions on the value of $z = \exp(-i\alpha)$.

In this units introduced above the infinitesimal generator for the CTM is given as

$$L_0 = \sum_j jH(j, j+1). \tag{2.11}$$

For L_0 the notion of periodic boundary conditions is not a sensible one. However, as we shall show below, the eigenstates of L_0 are localized so that the effect of the boundaries is negligible in the limit of an infinite chain. Acting with L_0 on the one magnon state (2.10) yields

$$L_0|z\rangle = \left(\frac{z}{w}\right)^{1/2} \frac{w}{w-z} \sum_j \left(\frac{wz-1}{w-z}\right)^{j-1} \times \left((j-1)\frac{w-z}{wz-1} + j\frac{wz-1}{w-z} + \cosh\lambda(2j-1)\right) S_j^-|0\rangle. \tag{2.12}$$

Comparing (2.10) and (2.12) (and neglecting boundary terms) we find that

$$L_0|z\rangle = \left(z\frac{\partial}{\partial z}\right)|z\rangle. \tag{2.13}$$

This identity shows that the eigenstates of the CTM are *Fourier transforms* of the Hamiltonian eigenstates—the integration being performed with respect to the rapidity z of the magnon: the eigenmode operators are given by

$$\Psi(l) = \oint \frac{dz}{z} z^{-l} B(z). \tag{2.14}$$

From (2.13) it follows that $|l\rangle \propto \Psi(l)|0\rangle$ are the normalized eigenstates of L_0 with eigenvalues l. Furthermore, analyticity requires $l+\frac{1}{2}$ to be an integer.

We now want to analyse the structure of these eigenstates in some detail: defining

$$|l\rangle = \sum_j a_j(l) S_j^-|0\rangle \tag{2.15}$$

one obtains

$$a_j(l) \propto \oint \frac{dz}{z^{l+1/2}} \frac{(wz-1)^{j-1}}{(w-z)^j} = -\oint d\zeta\, \zeta^{l-1/2} \frac{(w-\zeta)^{j-1}}{(w\zeta-1)^j}. \tag{2.16}$$

An interesting property of the states $|l\rangle$ is that—depending on the sign of l—they are different from the ground state on the positive or on the negative half-axis only, although they are derived from the spin-wave eigenstates of the Hamiltonian (2.10) which has a uniform expectation value of the magnetization $\langle S_j^z\rangle$: $a_j(l) = -a_{1-j}(-l)$ and $a_{j<1}(l) = 0$ for $l > 0$. Although this result is quite surprising in the context of this derivation, it is a consequence of the fact that the term $H(0, 1)$ is missing in L_0, and hence L_0 is a sum of operators acting on sites with $j \geq 1$ ($j < 1$) only.

The non-zero amplitudes for $l > 0$ are (see also [10]):

$$\begin{aligned} a_j(l = k+\tfrac{1}{2}) &= \sqrt{w^2-1}(-w)^{k-j} \sum_{m=0}^{k} \binom{j-1+m}{k}\binom{k}{m}(-1)^m w^{-2m} \qquad j \geq 1 \\ &= -\sqrt{w^2-1}\, w^{j-k-2} P_{j-1}^{(k-j+1,0)}\left(1-\frac{2}{w^2}\right) \qquad \text{for } j < k+2 \\ &= \sqrt{w^2-1}(-w)^{k-j} P_k^{(j-k-1,0)}\left(1-\frac{2}{w^2}\right) \qquad \text{for } j > k. \end{aligned} \tag{2.17}$$

($P_k^{(\alpha,\beta)}(x)$ are Jacobi polynomials.) We find that all of these states are localized around $j = k+1$ falling off exponentially for $j \gg k$, hence justifying our treatment of the boundaries. The states with the lowest eigenvalues are

$$|l = \tfrac{1}{2}\rangle = \sqrt{w^2-1} \sum_{j>0} (-1)^j w^{-j} S_j^- |0\rangle$$

and (2.18)

$$|l = -\tfrac{1}{2} = \sqrt{w^2-1} \sum_{j<1} (-1)^j w^{j-1} S_j^- |0\rangle.$$

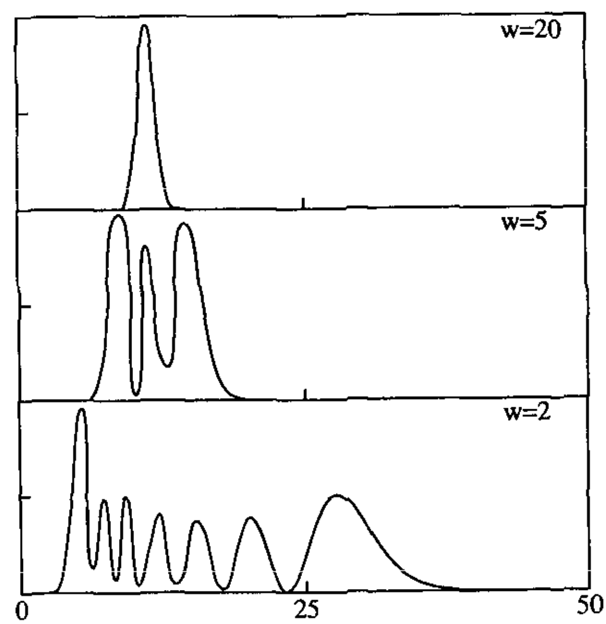

Figure 1. Amplitudes $|a_j(l)|^2$ of the eigenstate $|l = 21/2\rangle$ of the operator L_0 for different values of the anisotropy $2\Delta = (w^2+1)/w$. In the Ising limit ($w \to \infty$) the eigenstates are $a_j(l) = \delta_{j,l+1/2}$. As the anisotropy is reduced the eigenstates become broader—diverging as $\Delta \to 1$.

An example of a state with larger l is shown in figure 1: in the Ising limit $\Delta\to\infty$ the eigenstates of L_0 are localized overturned spins

$$|l\rangle = S^-_{l+1/2}|0\rangle \qquad \text{for } w\to\infty. \tag{2.19}$$

As Δ becomes smaller the state broadens, indicating the transition to the disordered state at the critical point $\Delta=1$.

Let us now consider states with two overturned spins, i.e. two interacting spin waves with rapidities z and ζ. From the algebraic Bethe ansatz we obtain the corresponding eigenstate of the Hamiltonian:

$$\begin{aligned}|z,\zeta\rangle &= B(z)B(\zeta)|0\rangle \\ &= \left(\frac{z}{w}\right)^{1/2}\left(\frac{\zeta}{w}\right)^{1/2}\frac{(w^2-1)^2}{(w-z)(w-\zeta)} \\ &\quad\times\sum_{j<k}\left\{\left(\frac{wz-1}{w-z}\right)^{j-1}\left(\frac{w\zeta-1}{w-\zeta}\right)^{k-1}\frac{\zeta-w^2z}{(z-\zeta)w}\right. \\ &\quad\left.+\left(\frac{wz-1}{w-z}\right)^{k-1}\left(\frac{w\zeta-1}{w-\zeta}\right)^{j-1}\frac{w^2\zeta-z}{(z-\zeta)w}\right\}S^-_jS^-_k|0\rangle. \end{aligned}\tag{2.20}$$

Again, it is straightforward to compute the action of L_0 on this state and comparison with the state (2.20) shows that

$$L_0|z,\zeta\rangle = \left(z\frac{\partial}{\partial z}+\zeta\frac{\partial}{\partial\zeta}\right)|z,\zeta\rangle. \tag{2.21}$$

This allows us to construct the eigenmodes of L_0 explicitly by Fourier transformation of the two-magnon states (2.20) with respect to the rapidities z and ζ. These transformations are to be performed independently. This constitutes the underlying reason for the previous observation that the two-body amplitudes can be written in terms of the one-body ones [10]: it merely reflects the nature of Bethe ansatz wavefunctions of the *XXZ* Hamiltonian, namely the fact that they can be written as suitable superpositions of single-particle plane wavefactors.

Note, however, that this procedure requires that the two rapidities be independent, i.e. z and ζ can be independently varied and always satisfy the Bethe ansatz equations. This is true for $|z|=|\zeta|=1$ (i.e. two interacting plane wave magnons). In the Ising limit $(w\to\infty)$ this gives (we assume $l\geqslant m$):

$$\Psi(l+\tfrac{1}{2})\Psi(m+\tfrac{1}{2})|0\rangle \propto \sum_{k=0}^{m-1} S^-_{m-j}S^-_{l+2+j}|0\rangle \tag{2.22}$$

for the eigenstate of L_0 corresponding to eigenvalue $l+m+1$. Note that for $m>1$ these states are not pure states with overturned spins at sites l and m as one might have concluded from (2.19). Furthermore, the two overturned spins are always separated. Hence the states (2.22) do not form a complete set.

However, the Bethe ansatz equations (2.7) allow for a different type of solution—so-called n-strings: sets of equally spaced rapidities α with the same real part

$$\alpha_k^{(n)} = \alpha_0^{(n)} + \mathrm{i}\lambda(n+1-2k) \qquad k=1,\ldots,n. \tag{2.23}$$

In the two-magnon sector the only possible configuration of this type is a two-string corresponding to a bound state of two magnons. The eigenstate of the Hamiltonian is again of the form (2.20) with $z=z^{(2)}w$ and $\zeta=z^{(2)}/w$ and in the Ising limit $(w\to\infty)$

these states reduce in fact to two neighbouring overturned spins moving together with centre of mass momentum being related to $z^{(2)}$ through

$$\exp(\mathrm{i}p^{(2)})=\frac{w^2z^{(2)}-1}{w^2-z^{(2)}}. \tag{2.24}$$

Substituting the expressions for z and ζ into (2.21) one finds

$$L_0|z^{(2)}w, z^{(2)}/w\rangle = z^{(2)}\frac{\partial}{\partial z^{(2)}}|z^{(2)}w, z^{(2)}/w\rangle. \tag{2.25}$$

Hence the corresponding eigenstate of L_0 is obtained by Fourier transformation with respect to $z^{(2)}$—the only free parameter left when one is restricted to the Hilbert space of the spin chain. In the Ising limit this gives

$$\Psi^{(2)}(l)|0\rangle \propto S_l^- S_{l+1}^-|0\rangle \tag{2.26}$$

for the eigenstate of L_0 with eigenvalue $l=1,2,\ldots$. Again, the exceptional nature of this eigenstate of L_0 [10] is just a reflection of the different nature of the corresponding Bethe ansatz eigenstate of the Hamiltonian.

We have now constructed all the eigenstates of L_0 in the sector with two overturned spins. The arguments used here can be applied to any state with a finite number of overturned arrows in the infinite system. For a system of finite length, the Fourier transform over the rapidities has to be replaced by a summation over discrete values. Analogous to the situation for the 2-strings we expect that these discrete values have to be chosen such that they satisfy the Bethe ansatz equations (2.7).

3. *N*-body states—intertwining of the CTM with the quantum inverse algebra

We can interpret and extend the results of the previous section to include any finite number of overturned arrows in an infinite volume. The relation between CTM eigenstates and Bethe ansatz states is expressed by an algebraic relation between the CTM and the elements of the QISM monodromy matrix (2.2) which follows directly from the Yang–Baxter relations [8].

The vertex matrix (2.1) can be shown to satisfy a set of Yang–Baxter equations (YBEs) of the form

$$[R_j(\alpha)R_{j+1}(\beta)]V_{j,j+1}(\alpha-\beta)=V_{j,j+1}(\alpha-\beta)[R_j(\beta)R_{j+1}(\alpha)]. \tag{3.1}$$

Here, the operator $V_{j,j+1}(\alpha-\beta)$ is also a vertex like R_j and R_{j+1}, but written as a two-spin operator rather than a 2×2 matrix of one-spin operators. Equation (3.1) is thus an operator relation for each component of the 2×2 matrices R_jR_{j+1}. For an infinitesimal argument, $V_{j,j+1}$ is related to the XXZ spin chain density (2.4) by

$$V_{j,j+1}(\varepsilon)=1+\varepsilon H(j,j+1)+\mathcal{O}(\varepsilon^2). \tag{3.2}$$

Expanding around $\alpha=\beta$ in (3.1) we obtain a useful infinitesimal form of the YBEs

$$[H(j,j+1), R_j(\alpha)R_{j+1}(\alpha)]=R_j'(\alpha)R_{j+1}(\alpha)-R_j(\alpha)R_{j+1}'(\alpha). \tag{3.3}$$

Multiplying both sides by j and summing over j we get a commutation relation between the infinitesimal CTM generator (2.11), and the monodromy matrix (2.2) [8, 11],

$$[L_0, \mathcal{T}(\alpha)] = \frac{\mathrm{d}}{\mathrm{d}\alpha} \mathcal{T}(\alpha). \tag{3.4}$$

It is easy to see that this commutator represents an algebraic statement of the results obtained in section 2. We only need to note that the Bethe ansatz states are obtained algebraically by (2.6), i.e. by repeated application of B-operators to the ferromagnetic ground state. Since $B(\alpha)$ is an element of the monodromy matrix, it satisfies (3.4)

$$[L_0, B(\alpha)] = \frac{\mathrm{d}}{\mathrm{d}\alpha} B(\alpha). \tag{3.5}$$

The Fourier transformed B-operators in (2.14) are thus the CTM eigenmode operators

$$[L_0, \Psi(l)] = l\Psi(l). \tag{3.6}$$

A CTM eigenstate containing any number of one-strings may be constructed from the ferromagnetic ground state by applying the eigenmode operators $\Psi(l)$. The construction of n-string Bethe ansatz states for $n \geqslant 2$ requires a continuation of the rapidity arguments of the B-operators to complex values. Our explicit consideration of the 2-string state in the previous section demonstrates that there is no difficulty in principle with this continuation, so long as we finally arrange the complex rapidity arguments into n-strings satisfying (2.23). (Otherwise, the corresponding coordinate space wavefunction will blow up exponentially for large site number.) For each value of n, one may define an n-string Bethe ansatz operator

$$\mathcal{B}^{(n)}(\alpha) = B(\alpha + \mathrm{i}(n-1)\lambda) B(\alpha + \mathrm{i}(n-3)\lambda) \ldots B(\alpha - \mathrm{i}(n-1)\lambda). \tag{3.7}$$

This compound operator also satisfies the shift property (3.5) with respect to the common real part α of the n-string rapidity

$$[L_0, \mathcal{B}^{(n)}(\alpha)] = \frac{\mathrm{d}}{\mathrm{d}\alpha} \mathcal{B}^{(n)}(\alpha). \tag{3.8}$$

Thus we may construct the corresponding n-string CTM eigenmode operators by Fourier transforming over the real part of the n-string rapidity

$$\Psi^{(n)}(l) = \oint \frac{\mathrm{d}z}{z} z^{-l} \hat{\mathcal{B}}^{(n)}(z) \tag{3.9}$$

where we have taken $\mathcal{B}^{(n)}(\alpha) = \hat{\mathcal{B}}^{(n)}(\mathrm{e}^{\mathrm{i}\alpha})$. The ferromagnetic ground state has an L_0 eigenvalue of 0, while all other eigenvalues are positive integers or half-integers. (Here we are considering a semi-infinite chain with the sum in (2.11) going from $j = 1$ to $j = \infty$.) In section 6 we will discuss the level-by-level counting of states and show that the CTM eigenstates exhibit the same degeneracy structure as a Heisenberg algebra of bosonic oscillators.

4. Filling the sea—the role of the Bethe ansatz equations

In the previous sections we have seen that the CTM eigenstates in the ferromagnetic regime may be constructed by Fourier transformation over the rapidity parameters in the Bethe ansatz for the corresponding XXZ Hamiltonian. In this case, the ferromagnetic ground state with all spins up is also the reference state upon which the Bethe ansatz is constructed. In the infinite volume limit with a finite number of down spins, the spectral parameters in the wavefunction form some number of n-strings, with the real rapidity of each n-string being independently variable. Thus, each real rapidity coordinate of an n-string may be independently Fourier transformed. A more complicated situation arises in the antiferromagnetic regime $\Delta < -1$, where the ground state has half of the spins turned down (assuming that the total number of spins N is even). The ground state and low-lying excited eigenstates of the Hamiltonian are described by highly non-trivial n-body Bethe ansatz wavefunctions, where n is approximately $N/2$. Furthermore, the rapidity parameters which appear in these wavefunctions are not all independent, but must be chosen to satisfy the Bethe ansatz equations. These may be regarded as periodic boundary conditions for the finite volume system. They limit the number of independent rapidity variables in the low-lying states to a few, corresponding to the number of physical 'particles' in the system. In the limit of infinite volume the Bethe ansatz equations reduce to integral equations which determine the density of modes in the filled Fermi–Dirac sea for the ground state and low-lying excited states.

We would now like to extend our discussion of CTM eigenstates to include the antiferromagnetic case. The prescription we obtained for the excitations above the ferromagnetic (empty sea) ground state, i.e. Fourier transformation over each rapidity variable in the wavefunction, clearly needs to be altered for the case of the filled sea. Fourier transformation over the rapidities of the vacuum modes would necessarily include values which did not satisfy the Bethe ansatz equations. Such states are not eigenstates of the Hamiltonian. The correct generalization of the previous discussion to the case of the filled sea may be motivated by the physical requirement that the CTM retain its interpretation as the lattice Lorentz boost operator. As we will see, the eigenstates of the CTM for the filled sea case are obtained from the eigenstates of the Hamiltonian by Fourier transforming over the rapidity variables which describe the location of the physical excitations (n-strings and holes) along the real rapidity axis. It is only these rapidity variables (and not those of the vacuum modes) which represent true degrees of freedom of the wavefunction. Thus, for example, the set of Hamiltonian eigenstates describing one physical particle excitation will be Fourier transformed over a single rapidity variable. In general, the number of physical particles in a state is easily determined by computing the energy eigenvalue and seeing how many rapidity variables it depends on.

The construction of the ground state and low-lying excited states for the XXZ Hamiltonian has been discussed in detail in several references [13]. In the infinite volume limit the discussion simplifies. Following Faddeev [14], we can summarize the relevant results as follows. The ground state is characterized by a density function $\rho_{\text{vac}}(\alpha)$ giving the density of modes in the filled Dirac sea. Similarly, an excited state with n holes at rapidities $\{\alpha_1, \ldots, \alpha_n\}$ has a density of sea modes given by a function of the form

$$\rho(\alpha; \{\alpha_i\}) = \rho_{\text{vac}}(\alpha) + \frac{1}{N} \sum_{l=1}^{n} \sigma(\alpha - \alpha_l). \tag{4.1}$$

The fact that the difference between the excited and ground state densities is given by a sum over terms associated with individual holes and that each of these terms depends only on the difference $\alpha - \alpha_l$ will be crucial in the construction of CTM eigenstates.

Using the density function $\rho_{\text{vac}}(\alpha)$ the filling of the sea may be concisely written in terms of the B-operators of the algebraic Bethe ansatz. Denoting the ferromagnetic (all spins up) reference state by $|0\rangle$, the physical ground state is [14]

$$|\Omega\rangle = \exp\left\{N \int \mathrm{d}\alpha\, \rho_{\mathrm{vac}}(\alpha) \ln B(\alpha)\right\}|0\rangle \tag{4.2}$$

The excited states with holes at $\alpha_1, \ldots, \alpha_n$ are then written

$$|\{\alpha_i\}\rangle = \prod_{i=1}^{n} \tilde{B}(\alpha_i)|\Omega\rangle \tag{4.3}$$

where the 'dressed' operators $\tilde{B}(\alpha_i)$ create the physical excitations, and are given explicitly by

$$\tilde{B}(\alpha_i) = \exp\left\{\int \mathrm{d}\alpha \sigma(\alpha - \alpha_i) \ln B(\alpha)\right\}. \tag{4.4}$$

In constructing CTM eigenstates, we will utilize the shift property of the dressed $\tilde{B}$ operators

$$[L_0, \tilde{B}(\alpha)] = \frac{\mathrm{d}}{\mathrm{d}\alpha} \tilde{B}(\alpha) \tag{4.5}$$

which follows from (3.5) and (4.4). The Fourier transformed operators are thus eigenmode operators of L_0

$$[L_0, \tilde{\Psi}(l)] = l\tilde{\Psi}(l) \tag{4.6}$$

where

$$\tilde{\Psi}(l) = \oint \frac{\mathrm{d}z}{z} z^{-l} \tilde{B}(z) \tag{4.7}$$

and we have set $z = \mathrm{e}^{\mathrm{i}\alpha}$.

The only other ingredient required for constructing the CTM eigenstates is a knowledge of the action of L_0 on the filled-sea ground state $|\Omega\rangle$. As we have discussed, this state contains no adjustable rapidity parameters. In the context of the lattice Lorentz group [8], $|\Omega\rangle$ is the Lorentz invariant vacuum state. Such physical reasoning suggests that $|\Omega\rangle$ is not only an eigenstate of the Hamiltonian, but also of L_0 (and hence, of the CTM). In fact, we will show in the next section that

$$L_0|\Omega\rangle = 0 \tag{4.8}$$

in other words, the ground state of the Hamiltonian is also an eigenstate of L_0 with zero eigenvalue. From the point of view of the explicit Bethe ansatz construction of $|\Omega\rangle$ this result is quite remarkable. An explicit verification of it would expose an intricate interplay between the action of the operator L_0 and the Bethe ansatz equations which determine the density of modes in the sea. We saw that, in an infinite volume with a finite number of overturned spins, the operator $\mathrm{e}^{\mathrm{i}\delta\alpha L_0}$ acts as a rapidity shift (boost) operator, i.e. it moves each mode in rapidity space by an equal infinitesimal amount $\delta\alpha$. However, the result (4.8) implies that, when acting on the filled sea, the shift induced on a mode by L_0 is not just $\delta\alpha$ but rather proportional to $\delta\alpha/\rho_{\mathrm{vac}}(\alpha)$. (Note that $\rho_{\mathrm{vac}}(\alpha)$ is not a constant, i.e. the density of modes in the sea is non-uniform in rapidity space, and therefore the shift induced by L_0 must also be non-uniform in order to leave the state unchanged.) The detailed combinatorics required to obtain (4.8) from the exact many-body ground state wavefunction have not been worked out.

Instead we have adopted a completely different approach to proving this result, using Schrödinger perturbation theory around the point $\Delta = -\infty$ to show that the result (4.8) is valid to all orders in $1/\Delta$.

5. Identity of the CTM and Hamiltonian ground state in the antiferromagnetic regime

In this section we shall use Schrödinger perturbation theory to construct the ground state of the XXZ Hamiltonian for large negative Δ (i.e. in the antiferromagnetic regime) and show that this state is also an eigenstate of L_0. We shall consider a spin chain with fixed boundaries—again the difference from periodic boundary conditions should be negligible in the thermodynamic limit.

To have a well defined problem we consider a chain on the half-axis with sites numbered from 1 to N. This procedure avoids the difficulty of having to deal with a large number of degenerate states (without imposing normal-ordering the ground state of L_0 is ferromagnetic (antiferromagnetic) on the negative (positive) half-axis). The final result is true for any numbering of the sites: if $|\Omega\rangle$ is an eigenstate of both L_0 and H (with sites labelled $1, \ldots, N$) then it is also an eigenstate of the operator $\tilde{L}_0 = L_0 - N/2H$ defined on the symmetric chain:

$$(L_0 - (N/2)H)|\Omega\rangle = (l_0 - (N/2)E_0)|\Omega\rangle. \tag{5.1}$$

At $\Delta = -\infty$ the system reduces to the Ising model and any configuration of spins is an eigenstate of the Hamiltonian. The ground state of both the Hamiltonian and L_0 is the Neél state

$$|\Omega^{(0)}\rangle = |\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow \ldots\rangle. \tag{5.2}$$

The boundary conditions at site 0 (for H) and at site N (for H and L_0) ensure that there is no degeneracy with the state with all the arrows turned.

The excited states with zero magnetization are conveniently labelled by the positions of pairs of spins that are flipped compared to the ground state, i.e. for a state with one pair of flipped spins at $(n, n+1)$:

$$|n\rangle = |\uparrow_1\downarrow_2\uparrow_3 \ldots \uparrow_{n-1}\uparrow_n\downarrow_{n+1}\downarrow_{n+2} \ldots\rangle. \tag{5.3}$$

Any state in the zero magnetization sector can be labelled in this fashion giving rise to a complete set of states of the form $|n_1, n_2, \ldots\rangle$ with $n_{k+1} - n_k \geq 2$.

The first-order correction to the ground states has non-vanishing amplitudes only for states of the form (5.3):

$$|\Omega^{(1)}\rangle = -\sum_n |n\rangle \frac{\langle n|V|\Omega^{(0)}\rangle}{W_n - W_0} \tag{5.4}$$

where

$$\begin{aligned} V_H &= \frac{1}{2\Delta} \sum_j S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ \\ V_L &= \frac{1}{2\Delta} \sum_j j(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) \end{aligned} \tag{5.5}$$

and W_0, W_n are eigenvalues of the Ising operators (we choose a different normalization of the operators here than in (2.4) to simplify the expressions—this does not affect the perturbative eigenstates obtained)

$$H^{(0)}=\sum_j (S_j^z S_{j+1}^z+\tfrac{1}{4}) \qquad L_0^{(0)}=\sum_j j(S_j^z S_{j+1}^z+\tfrac{1}{4}). \tag{5.6}$$

The matrix elements in (5.4) are easily obtained to be

$$\begin{aligned} H^{(0)}|n\rangle &= |n\rangle \qquad \langle n|V_H|\Omega^{(0)}\rangle=\frac{1}{2\Delta} \\ L_0^{(0)}|n\rangle &= n|n\rangle \qquad \langle n|V_L|\Omega^{(0)}\rangle=\frac{n}{2\Delta}. \end{aligned} \tag{5.7}$$

Hence, due to the cancellation of the factors n in the matrix element and the eigenvalue denominator for L_0 to first order in Δ^{-1} the ground states of the Hamiltonian and L_0 remain identical.

To second order the correction to the ground state amplitudes is

$$|\Omega^{(2)}\rangle=\sum_{n+1<n'}|nn'\rangle\sum_m\frac{\langle nn'|V|m\rangle\langle m|V|\Omega^{(0)}\rangle}{(W_{nn'}-W_0)(W_m-W_0)}. \tag{5.8}$$

Two cases have to be considered separately: for $|n-n'|>2$ we have

$$\begin{aligned} H^{(0)}|nn'\rangle &= 2|nn'\rangle \qquad \langle nn'|V_H|n\rangle=\frac{1}{2\Delta} \\ L_0^{(0)}|nn'\rangle &= (n+n')|nn'\rangle \qquad \langle nn'|V_L|n\rangle=\frac{n'}{2\Delta}. \end{aligned} \tag{5.9}$$

Hence, one obtains for the amplitude of the state $|nn'\rangle$ in the expansion of the ground state of L_0

$$\sum_m\frac{\langle nn'|V_L|m\rangle\langle m|V_L|0\rangle}{(l_{nn'}-l_0)(l_m-l_0)}=\frac{1}{n+n'}(\langle nn'|V_L|n\rangle+\langle nn'|V_L|n'\rangle)=\frac{1}{2\Delta}. \tag{5.10}$$

The same is obtained if one computes the amplitude for the Hamiltonian ground state.

The unperturbed eigenvalues are different if $|n-n'|=2$:

$$H^{(0)}|nn'\rangle=|nn'\rangle \qquad L_0^{(0)}|nn'\rangle=\tfrac{1}{2}(n+n')|nn'\rangle. \tag{5.11}$$

Again, the corresponding amplitudes in (5.8) are identical for H and L_0 due to the cancellation of factors $\propto (n+n')$ in the matrix elements and the denominators in the expression for L_0.

We have shown that up to second order in Δ^{-1} the ground states of the antiferromagnetic XXZ Hamiltonian and the corresponding lattice boost operator L_0 are identical. Going to higher orders in this perturbational analysis, the algebra becomes more and more tedious. It is quite straightforward, however, to convince oneself that the above result holds to any order in Δ^{-1}. While this was to be expected on physical grounds (the vacuum should remain unchanged under a boost) this result is by no means trivial in the context of the Bethe ansatz analysis (see the discussion in the previous section).

6. CTM eigenstates and bosonic oscillators

As mentioned in the introduction, one of our motivations for studying CTM eigenstates is to analyse their structure in terms of representation theory of Virasoro and Kac-Moody algebras. Although a full discussion of this subject would go beyond the intended scope of this paper, we will show here that the CTM eigenstates for the six-vertex model can be identified with those of an infinite set of bosonic oscillators a_n, $n\in\mathscr{Z}$ satisfying a Heisenberg algebra

$$[a_n, a_m] = n\delta_{n,-m} \tag{6.1}$$

and having the property

$$[a_n, L_0] = na_n \tag{6.2}$$

and

$$[a_n, N] = 0 \tag{6.3}$$

where N is the number operator which gives the number of down arrows (see below). Taking $a_n^\dagger = a_{-n}$, the operators a_n and $a_{-n}(n>0)$ are, respectively, annihilation and creation operators for the nth oscillator. The relation between the Heisenberg algebra (6.1) and the harmonic modes of a vibrating string is at the heart of the algebraic structure of string theory. The emergence of a similar structure in the study of CTMs is quite remarkable.

We work in the ferromagnetic regime and consider the eigenstates of L_0 with the spins numbered from 1 to ∞

$$L_0 = \sum_{j=1}^{\infty} jH(j, j+1). \tag{6.4}$$

Also, define the number operator which gives the number of overturned arrows in a state

$$N = \tfrac{1}{2}\sum_{n=1}^{\infty} (1-\sigma_j^z) \tag{6.5}$$

and consider the generating function

$$Q(q, z) = \mathrm{Tr}\{q^{L_0} z^N\} = \sum_{n=0}^{\infty} z^n\, \mathrm{Tr}_n q^{L_0} \tag{6.6}$$

$$\equiv \sum_{n=0}^{\infty} z^n Q_n(q) \tag{6.7}$$

where Tr_n is the trace over the subspace with n overturned spins. The function $Q(q, z)$ summarizes the information on the level structure and degeneracy of CTM eigenstates.

For $n=0$ there is only one state (all spins up) and we choose L_0 to have eigenvalue 0 on this state, i.e. $Q_0(q)=1$. For $n=1$ the eigenstates are given by (2.15). Taking the sum in (2.15) to go from $j=1$ to ∞, the state $|l\rangle$ vanishes for $l<\frac{1}{2}$. Thus the lowest eigenstate in the $n=1$ sector is $|\frac{1}{2}\rangle$, and the full set of states in that sector is given by

$$|l+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{N_l}}\Psi(l+\tfrac{1}{2})|0\rangle. \tag{6.8}$$

By choosing N_l such that the states $|l+\frac{1}{2}\rangle$ are orthonormal, we may identify these states with those of the first mode a_1 of the Heisenberg oscillator (6.1) acting on the lowest state $|\frac{1}{2}\rangle \equiv |\Omega_1\rangle$,

$$|l+\tfrac{1}{2}\rangle = (a_1^\dagger)^l |\Omega_1\rangle. \tag{6.9}$$

The contribution of these states to the generating function (6.6) is

$$Q_1(q) = \frac{q^{1/2}}{1-q}. \tag{6.10}$$

The absence of other Heisenberg states in the $n=1$ sector requires that

$$a_l^\dagger |\Omega_1\rangle = 0 \qquad l > 1. \tag{6.11}$$

For the $n=2$ states with two downturned spins, the lowest eigenstate of L_0 is the 2-string state with L_0 eigenvalue of 1,

$$|\Omega_2\rangle = \frac{1}{\sqrt{N^{(2)}}} \Psi^{(2)}(1)|0\rangle. \tag{6.12}$$

All of the other states in the $n=2$ sector can be obtained formally from $|\Omega_2\rangle$ by exciting the first two Heisenberg oscillators

$$(a_2^\dagger)^{l_2}(a_1^\dagger)^{l_1}|\Omega_2\rangle. \tag{6.13}$$

Again, the right number of states is obtained by assuming that the higher Heisenberg oscillators cannot be excited in the $n=2$ sector

$$a_l^\dagger |\Omega_2\rangle = 0 \qquad l > 2. \tag{6.14}$$

The easiest way to demonstrate these results is to consider the limit $\Delta \to \infty$ where, as we saw in section 2, the 2-string state with eigenvalue l reduces to two adjacent flipped spins at sites l and $l+1$, and the states with two 1-strings reduce to a simple linear combination of states with two flipped spins separated by at least one unflipped spin. (Thus adjacent flipped spins are always bound.) For finite Δ, we may appeal to the fact that the operator L_0 for any value of Δ may be obtained from the L_0 for any other value of Δ by a unitary transformation (since the eigenvalues remain unchanged). Thus, each L_0 eigenstate for finite Δ may be regarded as the unitary transformation of a particular spin state at $\Delta = \infty$. (For $n=2$, these states are formed by taking appropriate linear combinations of the states (2.22) with $l+m$ fixed.) The contribution of the $n=2$ sector to the generating function (6.6) is

$$Q_2(q) = \frac{q}{(1-q)(1-q^2)}. \tag{6.15}$$

For general values of n, it is easy to see that the lowest eigenstate in each sector is the n-string state with eigenvalue $n/2$

$$|\Omega_n\rangle = \frac{1}{\sqrt{N^{(n)}}} \Psi^{(n)}\left(\frac{n}{2}\right)|0\rangle. \tag{6.16}$$

The full set of states in each sector n exactly fills out the spectrum expected from the first n harmonic oscillators

$$(a_n^\dagger)^{l_n} \ldots (a_2^\dagger)^{l_2}(a_1^\dagger)^{l_1}|\Omega_n\rangle \tag{6.17}$$

which has an L_0 eigenvalue given by

$$\sum_{\nu=1}^{n} \nu l_\nu + \frac{n}{2}. \tag{6.18}$$

The contribution of sector n to the generating function is

$$Q_n(q) = \frac{q^{n/2}}{\prod_{l=1}^{n}(1-q^l)}. \tag{6.19}$$

Thus, the full generating function (6.6) is

$$Q(q,z) = \sum_{n=0}^{\infty} \frac{z^n q^{n/2}}{\prod_{l=1}^{n}(1-q^l)}. \tag{6.20}$$

If we set $z=1$ (i.e. sum q^{L_0} over all states, independent of the number of overturned spins), we obtain the following fermion–boson equivalence formula

$$Q(q,1) = \sum_{n=0}^{\infty} \frac{q^{n/2}}{\prod_{l=1}^{n}(1-q^l)} \tag{6.21}$$

$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-q^{n-1/2}}\right) = \prod_{n=1}^{\infty} (1+q^{n/2}). \tag{6.22}$$

The last result shows that if one ignores the distinction between different numbers of down arrows, the CTM eigenstates may also be regarded as a collection of free fermionic oscillators. This result can be obtained directly in the $\Delta = \infty$ limit by identifying each domain wall (= boundary between adjacent antiparallel arrows) as an occupied fermion level. Unlike the case of overturned arrows, the contribution of each domain wall to the L_0 eigenvalue is additive. In terms of the Heisenberg operators (6.1), one may construct a representation of the Virasoro algebra which contains the CTM generator L_0

$$L_n = \tfrac{1}{2}\!:\sum_{l=-\infty}^{\infty} \alpha_l \alpha_{n-l}\!:. \tag{6.23}$$

These operators satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}m(m^2-1)\delta_{m,-n}. \tag{6.24}$$

Although these results are obtained via a level-by-level counting of states, the problem of identifying the bosonic oscillator states (6.17) with the Fourier transformed Bethe states discussed in section 2 has not been resolved for general n. The complication involves the degeneracy of states at higher levels, which introduces an ambiguity in identifying boson states with spin states. An explicit operator construction of the Heisenberg oscillators in terms of the Fourier transformed n-string operators $\Psi^{(n)}(l)$ would be very useful. By standard methods, the Heisenberg algebra can be promoted, via a vertex operator construction, to a full representation of a level 1 SU(2) Kac–Moody algebra. The role of Kac–Moody algebras in the CTM formalism, though clearly indicated at the character level [4], still remains mysterious.

Acknowledgments

This work was supported by NSF under contract DMR-88-10541 and by the US Department of Energy under contract No. DEAS05-89ER40518.

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